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How to identify the physiological parameters and run the optimal race

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Abstract

This paper shows how a system of ordinary differential equations describing the evolution of the anaerobic energy, the oxygen uptake, the propulsive force and the velocity of a runner accurately describes pacing strategy. We find a protocol to identify the physiological parameters needed in the model using numerical simulations and time splits measurements for an 80 m and a 1600 m race. The velocity curve of the simulations is very close to the experimental one. This model could allow to study the influence of training and improving some specific parameters for the pacing strategy.

1. Introduction

The purpose of this study is to show that the model introduced in [1] accurately describes pacing strategy and to produce a way to compute the necessary physiological parameters given some time splits measurements on runners. Given a fixed distance to run, the idea is to determine the best pacing strategy and time performance.

Keller [6, 7] was the first to write a model to describe an optimal race. It is based on energy conservation and the fundamental principal of dynamics. With his simple model, and relying on constant maximal oxygen uptake (constant $\dot{V}O2$), he found that he could nevertheless reasonably predict the time of world records for distances bigger than 400 m. Several improvements of Keller's model have been introduced: the effect of fatigue [18, 8], the variation in maximal oxygen uptake [2], air resistance and altitude [13]. Other related works include [17, 9, 10, 12].

The recent work of [1] provides a system of equations taking into account a better description of physiology, based on the ideas of Keller, but encompassing a variable $\dot{V}O2$ and modeling the anaerobic energy using some ideas of the hydraulic analogy of Margaria, Morton [10]. Namely, we fix a distance D to run and we want to optimize on the time to run it. The model predicts instantaneously the velocity of the runner, his propulsive force, his oxygen uptake and his anaerobic energy. It relies on four physiological parameters: $\dot{V}O2_{max}$, the maximal oxygen uptake, e^0, the total available anaerobic energy, τ, the characteristic time of acceleration, or equivalently the internal resistance of the body and f_{max}, the maximal propulsive force. These parameters are not accessible by direct measurements. The aim of this study is to show that these four parameters can be determined from time splits of a run. From these estimated parameters, the model accurately describes real races. Additionally, the equations give access to $v(t)$, the velocity, $e(t)$ the anaerobic energy, $f(t)$ the propulsive force.

Let us now introduce our model. The first equation is the equation of motion, as in Keller's paper:

$$\frac{dv}{dt}(t) + \frac{v(t)}{\tau} = f(t) \tag{1.1}$$

where t is the time, $v(t)$ is the instantaneous velocity, $f(t)$ is the propulsive force per unit mass and v/τ is a resistive force per unit mass. We point out that other friction effects can be taken into account, such as air resistance or variations of altitude, but for simplicity of presentation, we do not include them here. Constraints have to be imposed; the force is controlled by the runner but it cannot exceed a maximal value f_{max} per unit mass, that is

$$0 \le f(t) \le f_{max}. \tag{1.2}$$

We now have to write the energy equation. In Keller's paper, the rate of oxygen uptake is assumed to be constant, called $\bar{\sigma}$. The energy balance is achieved by taking into account the creation of energy due to the oxygen uptake and the loss due to the work of the propulsive force:

$$\frac{de}{dt}(t) = \bar{\sigma} - f(t)v(t). \tag{1.3}$$

The aim is to

$$\text{Minimize the time } T \text{ given the distance } D = \int_0^T v(t)\,dt, \tag{1.4}$$

with the conditions:

$$v(0) = 0, \quad e(0) = e^0 \quad \text{under the constraint } e(t) \ge 0, \tag{1.5}$$

and solving (1.1)-(1.2)-(1.3). This optimal control problem leads to a race in three parts (for distances bigger than 400 m):

(1) the propulsive force is at its maximal value and the runner speeds up,

(2) constant velocity is reached for the biggest part of the race,

(3) the velocity and the propulsive force decrease on a zero energy branch.

These three parts have been proved to exist in this order in [1], where additionally, it is explained that Keller's model in fact describes the anaerobic energy through the accumulated oxygen deficit, rather than the aerobic energy as he had claimed. The velocity profiles obtained by simulating Keller's model display some inaccuracies with respect to reality, in particular since the end of the race is usually done by speeding up instead of slowing down. The idea of the new model introduced in [1] is to take into account the fact that the oxygen uptake varies along the race. In particular, a drop in the rate of oxygen uptake $\dot{V}O2$, at the end of the race, leads to an increase of velocity and propulsive force.

The experimental results of [4, 5] show that the rate of oxygen uptake $\dot{V}O2$ is not constant throughout the race but, on a 1500 m, rises steadily from an initial value of about $12\,\mathrm{ml\,min^{-1}\,kg^{-1}}$ to its maximum value $66\,\mathrm{ml\,min^{-1}\,kg^{-1}}$ over the first 20 to 40 seconds of the race and then drops to 60 in the last 200 m. Using the Respiratory Exchange Ratio which predicts that 1 L of oxygen in the body produces an energy of 20 kJ, the rate of oxygen uptake available for an effort can be converted into an energetic equivalent per unit of mass. This equivalent depends on the intensity of effort and can vary from 19 to 21, but 20 is a good average value [11]. We recall that $1\,\mathrm{J} = 1\,\mathrm{kg\,m^2\,s^{-2}}$ and that the experimental values of $\dot{V}O2$ are per minutes instead of per second. Hence the energetic equivalent per unit time and per kg of $\dot{V}O2$ is measured in $\mathrm{m}^2\,\mathrm{s}^{-3}$ and is thus $60/20 = 3$ times smaller than the experimental value of $\dot{V}O2$ measured in $\mathrm{ml\,min^{-1}\,kg^{-1}}$. We call this energetic equivalent σ. In [1], a function σ is built up in order to reproduce the experimental measurements of $\dot{V}O2$, but instead of writing it as a function of time, it is written as a function of the available anaerobic energy. The last part of the race, is the easiest one to understand: it takes into account that there are limitations when the energy supply is small. The experimental results of [4, 5] show that the rate of oxygen uptake falls slightly at the end of the race, dropping from 66 to about 60 over the last 200 m to 250 m. When the anaerobic energy e is too small, we quantify the drop in σ on the first line of (1.6). In the middle part of the race, the oxygen uptake is constant as chosen on the second line of (1.6). There is an initial phase of the race where σ rises linearly from its rest value to its maximal value. The value at which σ reaches $\bar{\sigma}$ is not a fixed fraction of e^0, but rather depends on the runner, in the sense that $e^0 - e$ is fixed. A hydraulic analogy is used in [1] to better identify the link between aerobic and anaerobic energy in the race: it is assumed that the aerobic energy is of infinite capacity and flows at a maximal rate of $\bar{\sigma}$ into the anaerobic container which is of finite capacity. The flow from the aerobic container is proportional to the difference of fluid heights in the containers. Therefore, the difference $e^0 - e$ is related to the non dimensionalized height of the container and, when it reaches the critical value γ_2, the flow becomes constant at rate $\bar{\sigma}$. This value γ_2 is related to the height at which the aerobic container is connected to the anaerobic one, but not to the full volume of the anaerobic container.

This leads to the following function introduced in [1] depending on the anaerobic energy $e(t)$:

$$\sigma(e) = \begin{cases} \bar{\sigma}\frac{e}{e^0\gamma_1} + \sigma_f(1 - \frac{e}{e^0\gamma_1}) & \text{if } \frac{e}{e^0} < \gamma_1 \\ \bar{\sigma} & \text{if } \gamma_1 \le \frac{e}{e^0} \text{ and } \gamma_2 \le e^0 - e \\ \sigma_r + (\bar{\sigma} - \sigma_r)\frac{e^0 - e}{\gamma_2} & \text{if } 0 \le e^0 - e \le \gamma_2, \end{cases} \tag{1.6}$$

where σ_r is the rest value, σ_f the final value and $\bar{\sigma}$ the maximal value.

Numerically, we choose $\sigma_r = 6$, $\sigma_f = 20$, $\bar{\sigma} = 22$, $\gamma_1 = 0.15$ and $\gamma_2 = 566$, which is close to $e^0/4$, given the values of e^0 used. In order to avoid piecewise linear functions, we make a regularization. Figure 1.1 plots $\sigma(e(t))$ as a function of time. Since the energy $e(t)$ is a decreasing function of time (roughly linearly), at $t = 0$, $e = e^0$ and $\sigma = \sigma_r$ and at the final time, $e = 0$, and $\sigma = \sigma_f$. The plot of $\sigma(e(t))$ looks in reverse order with respect to $\sigma(e)$.

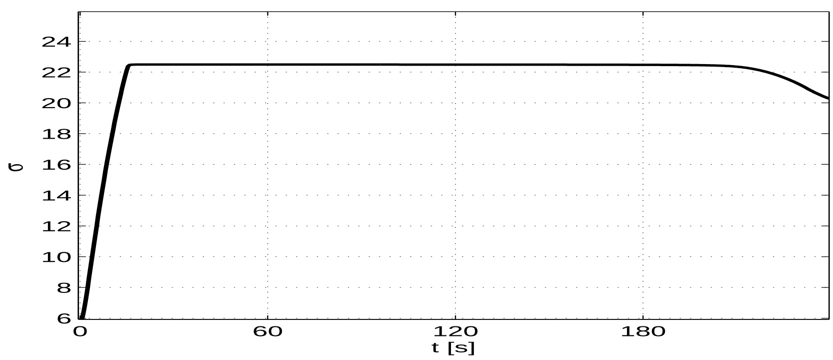

FIGURE 1.1. Energy equivalent of oxygen uptake (measured in m^2s^{-3}) vs time for a 1500 m.

This allows us to write an equation for the anaerobic energy which takes into account the work of the propulsive force and the variations of oxygen uptake:

$$\frac{de}{dt}(t) = \sigma(e(t)) - f(t)v(t). \tag{1.7}$$

Equations (1.1)-(1.2)-(1.5)-(1.6)-(1.7) with the aim (1.4) is a well defined optimal control problem. We solve it with the optimal control solver Bocop [3] to find the numerical optimal solution. More details are given in [1]. For a 1500 m, this yields the function $\sigma(e)$ of Figure 1.1, which is consistent with the experimental results of [4, 5].

The aim of this study is to determine the four parameters of a runner: τ, f_{max}, e^0, $\bar{\sigma}$. To this end, we have to find the optimal race described by the model which is the closest to the experimental race performed by a runner. The dashed lines in Figures 3.1, 3.2, 3.3 and 3.4 present the computed velocity curve for four different runners. It is made of 3 parts:

- an initial acceleration with the propulsive force at its maximal value,
- an almost constant speed,
- a speed up at the end of the race where the maximal propulsive force is put back.

We introduce an experimental protocol which is described below to determine the physiological parameters.

2. **Experimental Method**

2.1. **Protocol**

Two races of 80 m and 1600 m were performed by 4 experimented runners, 3 males and 1 female aged 21-22. They ran alone and had been instructed to perform their optimal race, without any instruction during the race. The races were filmed and analyzed to determine the instantaneous speed. All subjects consented to participate in the experiment upon being informed of the purpose of the study and the protocol, and provided written informed consent, which was approved by the local ethics committee.

2.1.1. *Experimental set up*

Position markers were set up along the 80 m track every 2 meters. A video camera (Sony handycam, HDR-CX350) was placed in the center of the athletic track on a foot that allowed horizontal rotation. This rotation allowed the cameraman to film the entire race by targeting at the runner. A piece of paper (10×30 cm) was placed on the left size of the runner's chest. We consider the runner is overcoming a marker when the camera, the piece of paper and the marker are aligned. The 80 m race movie was analysed in order to determine the speed data. After the analysis, the trackers gave 40 time measurements and the speed could be computed by: $v_i = \frac{t_i - t_{i-1}}{d_0}$ with $d_0 = 2$ m.

For the 1600 m race, in order to register precise speed data with a relative quick analysis, the analysis was divided into three parts. The speed data were registered with a video at the beginning and at the end of the race, where the speed fluctuations are the most significant. So, for the first 40 m and the final 40 m, the speed data were collected with the same process as for the 80 m race. We used a GARMIN watch to register the data of the middle of the race.

These two analysis led us to curves showing the speed as a function of time for the 80 m race and the 1600 m race.

2.2. **Description of the computational process**

The aim is to compute the four physiological parameters of a runner, τ, f_{max}, e^0, $\bar{\sigma}$ from (1.1)-(1.2)-(1.5)-(1.6)-(1.7) with the aim (1.4).

- **Step 1:** Construction of admissible parameters for a runner
  (1) Using the experimental 80 m race, one can determine admissible parameters (f_{max}, τ). Indeed, in this case, the model predicts the velocity profile $v(t) = f_{max}\tau(1-\exp(-\frac{t}{\tau}))$. To have an estimate of the parameters (f_{max}, τ), one has to extract from the 80 m data two observations points $(t_1^{exp}, v^{exp}(t_1))$ and $(t_2^{exp}, v^{exp}(t_2))$. One can evaluate analytical expression of $v(t)$ at these points: it results in a two equations system with two unknowns. One can repeat this for other 80 m observations points, and therefore get multiple admissible parameters (f_{max}, τ).
  (2) In order to get a first approximation of $\bar{\sigma}$, one can use the Cooper test which consists in running for 12 min. The distance covered and the age provide a reasonnable estimate of $\bar{\sigma}$. Moreover, it is possible to make some assumptions based on the physiology of the observed runner. Then one can take several values uniformly distributed around the first guess of $\bar{\sigma}$.
  (3) It is then possible to compute an estimate for e^0, using equation (1.7) in the middle part of the race where the velocity is almost constant and σ equal to its maximal value $\bar{\sigma}$. This formula depends on $\bar{\sigma}$, τ and the starting time t_{beg} and the final time t_{end} of this constant speed phase. For all previous admissible parameters, and all estimates (t_1, t_2) of (t_{beg}, t_{end}), one can compute multiple admissible initial anaerobic energy through $e^0 = \frac{1}{1-\gamma_1}[\bar{\sigma}(t_2 - t_1) + \gamma_2 - \frac{1}{2}(v^2(t_2) - v^2(t_1)) - \int_{t_1}^{t_2} \frac{v^2(s)}{\tau} ds]$.

- **Step 2:** Determination of the best parameters among admissible ones. From step 1, one can get a list of admissible quadruplets $(f_{max}, \tau, \bar{\sigma}, e^0)$.
  (1) Given this list of admissible parameters, one has to launch the computation of the model's optimal control problem for each quadruplets as inputs. For each simulation, outputs of matter are velocity values $v_{num}(t_i)$ $i = 1, 2..N^{num}$, where N^{num} depends on the numerical discretisation. It is also necessary to store a one to one mapping, between vector of velocity values and associated quadruplets of parameters. Computation of all the optimal races are performed with the BOCOP software [3] for each quadruplet included in the intervals made in the first step. An average of 200 computations must be made in order to get precise results.
  (2) The criterion of selection is based on the least square method for the computed speed vectors $\vec{v}_{num}$ and the 1600 m observation $\vec{v}_{exp}$. More precisely, the runner's parameters would be those which minimize : $\sum_{i=1}^{N^{exp}}(v_{num}(i) - v_{exp}(i))^2 + \nu \mid T_{exp} - T_{num} \mid$. The second term penalizes numerical races with a final time different from the observed one.

As a summary, we see that the 80 m race is used to get rough estimates for τ and f_{max}. An approximate value of $\dot{V}O2_{max}$ is required and an approximate value of e^0 is computed. Then more precise values of the parameters are obtained thanks to the time splits for the 1600 m race using a least square method.

3. **Results**

The computation of the parameters is summarized in Table 3.1, where T_f is the final time for the 1600 m race.

Runner	τ	f_{max}	e^0	σ	T_f
A	0.87	8.80	1250	22.5	353.6
B	0.92	7.64	1456	23	339.2
C	0.82	9.52	3702	23.5	309.5
D	0.75	8.49	2500	17	399.6

Table 3.1. Parameters obtained using our protocol for the four runners

Then the optimal race solving (1.1)-(1.2)-(1.5)-(1.6)-(1.7) with the aim (1.4) using the parameters of Table 3.1 is computed. This computation is illustrated with the dashed line in Figures 3.1, 3.2, 3.3 and 3.4, while the solid line are the experimental points. We can see that the optimal run computed is close to the experimental one. Only runner D ends the race with the optimal acceleration and slight decrease. Runner A has a much stronger acceleration in the end, probably due to psychological factors (being able to push more strongly close to the finish line). As for runners B and C, it may be that their decrease in $\dot{V}O2$ is stronger than the one we have modelled. Indeed, γ_1, γ_2 and σ_f are fixed parameters in the simulations, but in a more complete model, could be used as free parameters which could be optimized in the least square method.

4. Discussion

Once the physiological parameters of the runner are determined, the main interest of the model is to predict how an improvement of a specific parameter through training will change the race and final time and to quantify the improvement with respect to the intended gain. The pacing strategy is indeed an important topic of research [15, 16, 5, 14].

Thanks to our model, the optimal pacing strategy can be computed on any distance for any runner. For 800 m, the velocity curve provides a strong acceleration, followed by a slowing down, and a speed up again. Thus the first lap is indeed run quicker than the second one. For longer races, the beginning and end of the race are significantly faster than the middle part, due to the velocity peaks at the beginning and end of the race, with an almost even pace in the middle part. This confirms the finding of [15, 16].

In the next figures, we vary by around 20% the different parameters σ, e^0, f_{max} and τ to see the effect on the race. The reference parameters are that of Runner A.

4.1. Variation in the maximal oxygen uptake

As we had mentioned before, σ is related to $\dot{V}O2$ by a factor 3, so varying σ between 18.5 and 25.5 amounts to varying $\dot{V}O2$ between 55 and 76. We keep all the other parameters fixed, impose a time to run and plot the velocity according to the distance in Figure 4.1. The runner with higher $\dot{V}O2$ is the one who runs the longest. One can see that $\dot{V}O2$ has an impact on the mean velocity during the race and the velocity at the final sprint. For a higher $\dot{V}O2$, the runner can maintain a higher mean velocity, but also can maintain it much longer.

The beginning of the race is not affected by a higher σ.

4.2. Variation in the total anaerobic energy

A similar experiment is performed with a variation of the initial anaerobic energy e^0 in Figure 4.2. A higher energy implies a higher mean velocity. The energy has no impact on the beginning of the race, and not much on the distance at which they speed up again at the end. The runner with higher energy can put back a bigger force and higher velocity at the end of the race and speed up for a longer time.

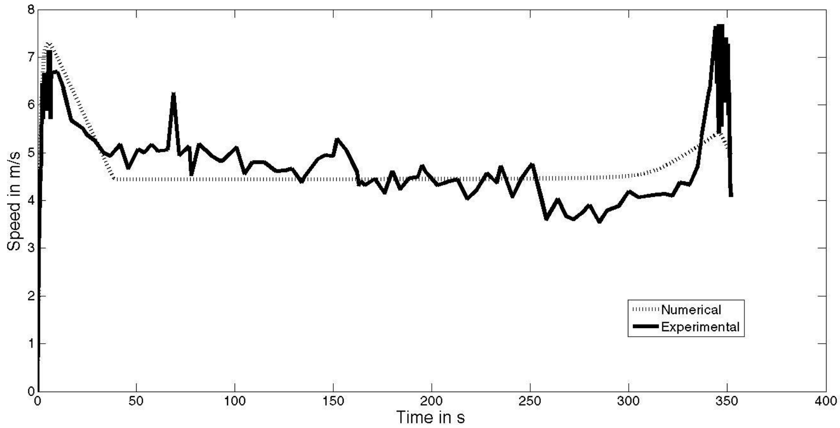

Figure 3.1. Velocity w.r.t. time: **Runner A**

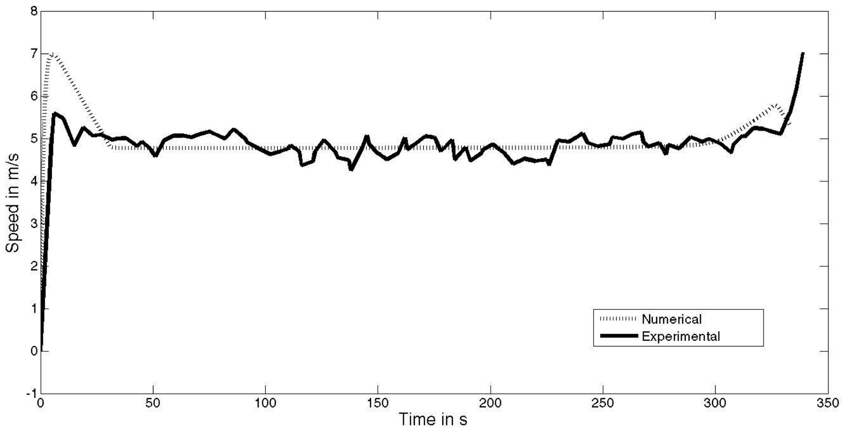

Figure 3.2. Velocity w.r.t. time: **Runner B**

4.3. Variation in the propulsive force

Concerning τ and f_{max}, we first recall that from the velocity equation, the peak velocity v_{peak} is determined by $v_{peak} = f_{max}\tau$. So increasing the propulsive force f_{max} or decreasing the friction, which is increasing τ leads to an increase on the peak velocity. In particular, increasing f_{max} produces an increase of the peak velocity at the beginning and at the end of the race, without modifying the middle part of the race. Indeed, in the middle part of the race, the force is at an intermediate value less than f_{max}. Increasing τ also has an impact on the middle part because as we have seen, τ has an impact on the energy consumed in the middle part of the race.

What seems significant is to keep the peak velocity fixed, which means varying both τ and f_{max} but keeping the product constant, as illustrated in Figure 4.3. A smaller τ implies a lower mean velocity along the race, and it is more difficult to put back a high velocity at the end of the race. A bigger τ means that v_{peak} can be maintained longer.

In order to increase the power output at the end of the race, one has to increase the anaerobic energy or the propulsive force.

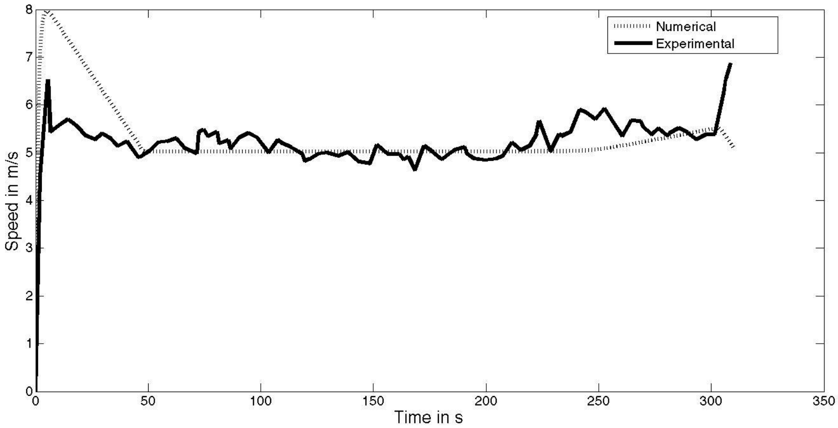

Figure 3.3. Velocity w.r.t. time: **Runner C**

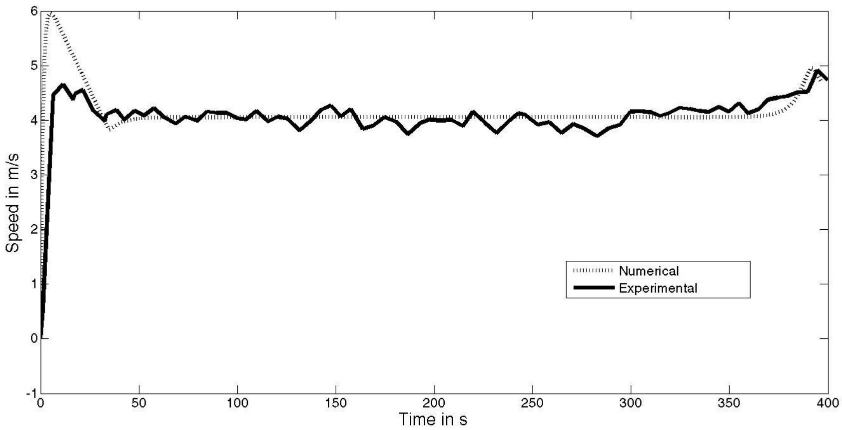

Figure 3.4. Velocity w.r.t. time: **Runner D**

5. **Conclusion**

Given a fixed distance to run, the model of [1] encompasses the optimal control of anaerobic energy, oxygen uptake, propulsive force and velocity to produce the computation of the best pacing strategy and time performance. The difficulty for practical applications is to identify the parameters. This includes the oxygen uptake curve provided by [5] according to the distance, and four physiological constants: the maximal oxygen uptake, the maximal available propulsive force, the available anaerobic energy and a friction factor, also related to the economy of a runner. Given time splits measurements on an 80 m and a 1600 m race, we introduce a protocol to identify these parameters. The 80 m race is used to get rough estimates for τ and f_{max}. Then more precise values of the parameters are obtained thanks to the time splits for the 1600 m race using a least square method. An approximate value of $\dot{V}O2_{max}$ is required.

This allows us to produce the optimal pacing curve for any distance and predict the possible improvements of a runner according to possible improvements of his physiology. The findings are consistent with the literature, and in particular [15, 16, 5, 14]. This model could potentially be useful for the training of athletes to analyze better their weaknesses and possibilities of performance improvements.

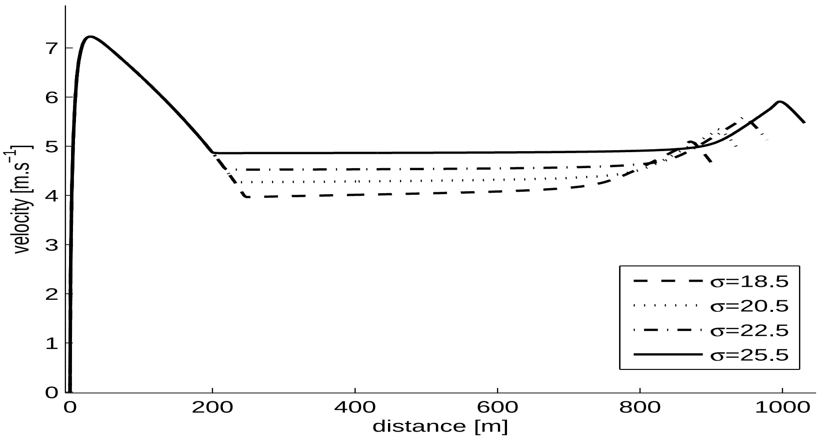

Figure 4.1. Velocity vs distance as the maximal oxygen uptake varies

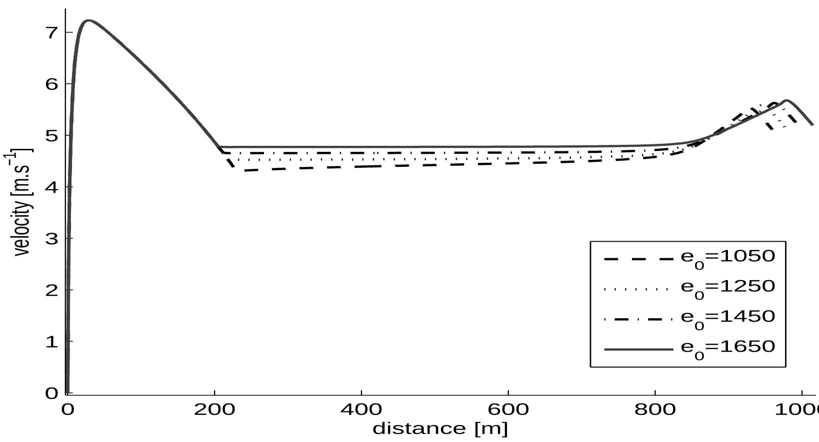

Figure 4.2. Velocity vs distance as the total anaerobic energy varies

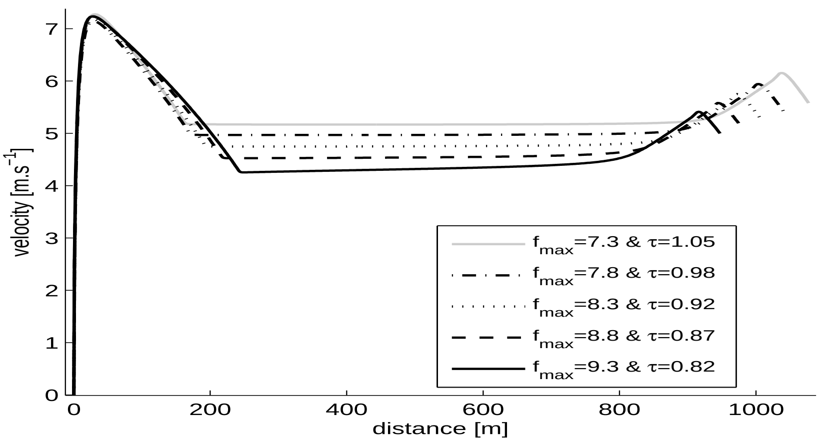

Figure 4.3. Velocity vs distance as the maximal propulsive force varies

References

[1] A. Aftalion and J.F. Bonnans. Optimization of running strategies based on anaerobic energy and variations of velocity. *SIAM Journal on Applied Mathematics*, 74(5):1615–1636, 2014.

[2] H. Behncke. A mathematical model for the force and energetics in competitive running. *Journal of mathematical biology*, 31(8):853–878, 1993.

[3] F. Bonnans, V. Grelard, and P. Martinon. Bocop, the optimal control solver, open source toolbox for optimal control problems. *URL http://bocop.org*, 2011.

[4] C. Hanon, J-M. Leveque, C. Thomas, and L. Vivier. Pacing strategy and $\dot{VO}2$ kinetics during a 1500-m race. *International journal of sports medicine*, 29(03):206–211, 2008.

[5] C. Hanon and C. Thomas. Effects of optimal pacing strategies for 400-, 800-, and 1500-m races on the $\dot{VO}2$ response. *Journal of sports sciences*, 29(9):905–912, 2011.

[6] J. B. Keller. A theory of competitive running. *Physics today*, page 43, 1973.

[7] J. B. Keller. Optimal velocity in a race. *American Mathematical Monthly*, pages 474–480, 1974.

[8] R. Maroński. Minimum-time running and swimming: an optimal control approach. *Journal of biomechanics*, 29(2):245–249, 1996.

[9] R. H. Morton. A 3-parameter critical power model. *Ergonomics*, 39(4):611–619, 1996.

[10] R. H. Morton. The critical power and related whole-body bioenergetic models. *European journal of applied physiology*, 96(4):339–354, 2006.

[11] F. Peronnet and D. Massicote. Table of nonprotein respiratory quotient: an update. *Can J Sport Sci*, 9:16–23, 1991.

[12] A. B. Pitcher. Optimal strategies for a two-runner model of middle-distance running. *SIAM Journal on Applied Mathematics*, 70(4):1032–1046, 2009.

[13] M. Quinn. The effects of wind and altitude in the 400-m sprint. *Journal of sports sciences*, 22(11-12):1073–1081, 2004.

[14] C. Thiel, C. Foster, W. Banzer, and J. De Koning. Pacing in olympic track races: competitive tactics versus best performance strategy. *Journal of sports sciences*, 30(11):1107–1115, 2012.

[15] R. Tucker, M. I. Lambert, and T. D. Noakes. An analysis of pacing strategies during men's world-record performances in track athletics. *International journal of sports physiology and performance*, 1(3):233, 2006.

[16] R. Tucker and T. D Noakes. The physiological regulation of pacing strategy during exercise: a critical review. *British journal of sports medicine*, 43(6):e1–e1, 2009.

[17] A.J. Ward-Smith. A mathematical theory of running, based on the first law of thermodynamics, and its application to the performance of world-class athletes. *Journal of biomechanics*, 18(5):337–349, 1985.

[18] W. Woodside. The optimal strategy for running a race (a mathematical model for world records from 50 m to 275 km). *Mathematical and computer modelling*, 15(10):1–12, 1991.